Press Release

Keio University

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Keio University

A Unique Pair of Triangles
—Proof of a Simple Theorem Using Abstract Modern Mathematics—

Yoshinosuke Hirakawa and Hideki Matsumura, graduate students of the Keio Institute of Pure and Applied Sciences (KiPAS) Arithmetic Geometry and Number Theory Group at the Keio University Faculty of Science and Technology, have proven a new theorem that states there is only one pair (up to similitude) of an isosceles triangle and a right triangle for which the lengths of all its sides are integers and which have the same perimeter and the same area.

The lengths of lines and areas of figures are basic geometrical quantities that are indispensable when measuring everything around us. For example, a familiar figure from textbooks is a right triangle with sides of lengths 3, 4, and 5 respectively. Moreover, an important question that has been studied since the time of ancient Greece is how many right triangles there are for which the lengths of all sides are integers. One field of modern mathematics which has greatly developed in the twentieth century under the influence of this tradition is arithmetic geometry.

The above theorem was proved by applying the "theory of p-adic Abelian integrals" and "descent of rational points" in arithmetic geometry. It is rare that highly abstract modern mathematics has applications to such familiar objects.

The above research is to be published under the title of "A unique pair of triangles" in the "Journal of Number Theory." (An electronic version has already been released as an "article in press" on August 24, 2018.)

1. Main Points of Research

- Triangles for which the lengths of all sides are integers have been studied since the time of ancient Greece. In this research, a new theorem was discovered and proven.
- Although this theorem appears to be elementary, it was proven using relatively modern methods of arithmetic geometry developed around the end of the twentieth century.
- It is rare that highly abstract modern mathematics has applications to such familiar objects.

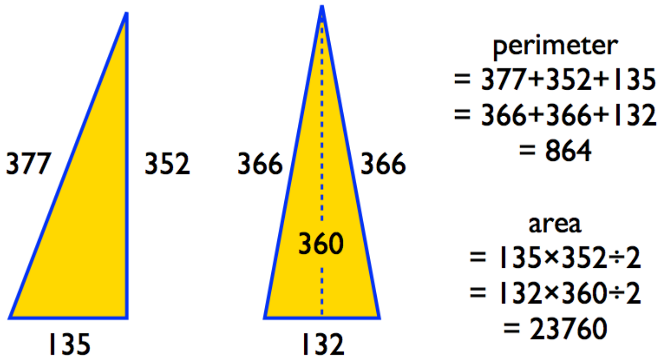
2. Background of Research

The lengths of lines and areas of figures are basic geometrical quantities that are indispensable when measuring everything around us. For example, a familiar figure from textbooks is a right triangle with sides of lengths 3, 4, and 5 respectively. Moreover, an important question that has been studied since the time of ancient Greece is how many right triangles there are for which the lengths of all sides are integers. Similarly, it is probable that the question of how many pairs of right triangles with integer lengths for all sides have the same perimeter and the same area had also been studied.

All of these questions can be rephrased in terms of the problem of determining the set of rational points on an algebraic curve of genus 0 (*1, 2). It has been known since at least the seventeenth century with the birth of coordinate geometry that this problem could be solved via a method called "rational uniformization." However, even today, any uniform method of solving equations like the Fermat equations ($x^n+y^n = 1$) which are restated in terms of a problem that determines the set of rational points on an algebraic curve of a genus greater than 1 remains unknown. Arithmetic geometry greatly developed in the twentieth century as one area of modern mathematics that took up the challenge of solving such difficult problems.

3. Content of Research and Results

In this research, Hirakawa and Matsumura proved a new theorem stating that there is only one pair (up to similitude) of an isosceles triangle and a right triangle for which the lengths of all its sides are integers and which have the same perimeter and the same area. The following figure shows this unique pair.



In the proof of this theorem, all the pairs of triangles in question were parameterized by an algebraic curve of genus 2, reducing the original problem to the separate problem of determining the set of rational points on a specific algebraic curve of genus 2. Although it is known that there are only a finite number of rational points on such an algebraic curve, in order to completely determine the set of rational points, more sophisticated techniques are required.

By applying the so called Chabauty-Coleman method based on the analytic theory of p-adic Abelian integrals, it was proven that there are only ten rational points on the above algebraic curve. Among the ten rational points, eight correspond to pairs of "collapsed" triangles for which the length of the sides are either zero or negative. The remaining two points correspond to the pair of triangles in the above figure. On the other hand, the main requirement for implementing the Chabauty-Coleman method is that a certain integer called the Mordell-Weil rank (*3) of the algebraic curve must be smaller than the genus. This requirement was established by proving that the Mordell-Weil rank is 1 using a cohomological method of descent argument.

Although this problem may have been considered from as early as the time of ancient Greece, the Chabauty-Coleman method and the descent argument used in the proof of this theorem are both relatively new methods with developments only beginning in the 1980s. This striking contrast between a simple ancient problem and sophisticated and contemporary techniques to bring about a solution highlights the beauty of modern mathematics

Details of Journal Article

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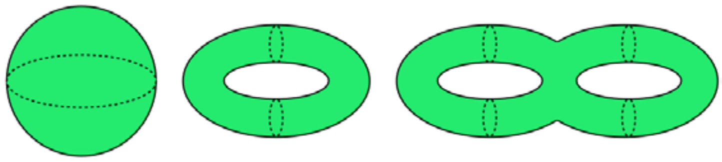
doi:10.1016/j.jnt.2018.07.007

<Glossary>

*1 Algebraic curve and genus: A figure defined as the set of solutions of an algebraic equation is called an algebraic variety. In particular, a one-dimensional algebraic variety is called an algebraic curve. For example, a straight line (y = x), a circumference (x^2+y^2 = 1), a parabola (y = x^2), a hyperbola (x^2 = y^2+1), etc., on the xy-plane are all algebraic curves. On the other hand, algebraic curves defined as sets of complex number solutions of equations are identified with a finite number of "swimming rings" connected to each other. (Contrary to its designation as a curve, it is identified with a surface.) The number of the rings is the genus of the curve. For example, the genera of the above straight line, circumference, parabola, and hyperbola are all 0. The genera of the algebraic curves defined by the Fermat equations, x^n+y^n = 1, are (n-1)(n-2)/2 respectively.



*2 Rational points: Among the points of an algebraic variety, a point that corresponds to a rational number solution of the algebraic equation is called a rational point. It could also be said to be a point for which all of its coordinates are rational numbers.

*3 Mordell-Weil rank: An algebraic curve with rational points can be embedded into a higher dimensional algebraic variety called the Jacobian variety of an algebraic curve via a standard method. There is an integer called the Mordell-Weil rank of a Jacobian variety, which measures the size of the set of rational points on this Jacobian variety. For example, the Jacobian variety has a finite number of rational points if and only if the Mordell-Weil rank is 0.

*4. Descent argument: It is often difficult to determine the Mordell-Weil rank of a Jacobian variety. In practice, however, it is often sufficient to give an upper bound (an integer not smaller than the Mordell-Weil rank). Based on this requirement, the Jacobian variety is embedded into "containers" called the cohomology group and Selmer group that is easier to estimate. A method giving an upper bound of the Mordell-Weil rank by estimating the sizes of these "containers" is called the descent argument.

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